

Kähler MMP and Transcendental BPF

X compact Kähler manifold / var

ω Kähler form, i.e. closed real $(1,1)$ -form

s.t. $\omega(\cdot, J\cdot) = g(\cdot, \cdot)$

or locally $i\partial\bar{\partial}(h)$, h strictly psh fn.

Kähler - Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} \omega(t) = - \underbrace{\text{Ric}}_{\partial\bar{\partial}}(\omega(t)) & t \rightarrow +\infty \\ \omega(0) = \omega \end{cases}$$

(normalized flow)

$$\begin{aligned} [\omega(t)] &= [\omega] - t \underbrace{C_1(X)} \\ &= [\omega] + t K_X \end{aligned}$$

Parabolic Monge-Ampère Eq

limit solution $\omega(t)$

Kähler-Einstein metric

$$K_X + t\omega$$

Analytic MMP

[Song-Tian 09 17]

$$X \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_n \quad K_{X_n} \text{ nef}$$

$$\searrow \quad \swarrow$$

Z_1

$K_X + t_1 \omega$ nef but not Kähler

Question: $\alpha = K_X + \omega$ nef $\stackrel{?}{\implies}$ semi-ample, i.e.

$$\exists f: X \rightarrow Y \quad \text{proj}$$

$\uparrow$
normal Kähler

ω_Y Kähler s.t.

$$\alpha \equiv f^* \omega_Y \quad \text{in } H^{1,1}$$

Difficulty:

- ① Construct the morphism $f: X \rightarrow Y$
- ② Show f is projective $\exists L$ on X s.t. L is f -ample
- ③ Show Y, ω_Y is Kähler

$$\text{Current } \beta_{X'} \rightarrow \beta_X \\ X' \dashrightarrow X$$

Main result: (Hacon-McD26)

Thm (BPF)_n: $(X, B + \beta)$ $\leftarrow$ b-(1,1) form cpt n-dim gklt Kähler pair

such that $\alpha = K_X + B + \beta_X$ nef

$B + \beta_X$ is (modified) big

$\Rightarrow \alpha$ is semi-ample: $\exists$ Moishezon contr.

$$f: X \rightarrow Y \quad \text{s.t.} \\ \text{bir } \uparrow \text{ proj } \nearrow \\ X' \rightarrow X$$

(1) Y cpt Kähler space

(2) $\alpha \equiv f^* \gamma$ γ Kähler class

Moreover, if X is strongly $\mathbb{Q}$ -factorial $\Rightarrow f$ proj.

X klt sing

X Moishezon

+ Kähler

$\Downarrow$

X proj

(K_X)

L

reflexive rank one

L om

line bundle

[Nam02]
Namikawa

Algebraic

$H^{1,1} \cap H^2(X, \mathbb{Z})$ Analytic

$$NS(X) := \text{Pic}(X) / \equiv_{\text{num}}$$

$$N_{\text{an}}^1(X) := H_{BC}^{1,1}(X)$$

$\uparrow$
Bott-Chern cohomology
(1,1) ^{current} forms with local potentials
/ $\text{d}d^c u$

$$NE(X) \subseteq N_1(X)$$

$$NE(X) \subseteq NA(X) \subseteq N_1(X)$$

$\uparrow$
[C] ^{positive} real closed $(n-1, n-1)$ -current

nef: ample $\sim$
big: $A + E$

Kähler

$$T \geq W$$

$$\mu^* T \sim W + E$$

pseudo-eff:

$$T \geq 0$$

positive current

$$[E] = \frac{i}{\pi} \partial \bar{\partial} (\log |f|) \quad \text{div}(f) = E$$

[Höring-Peternell 16] 3-fld.

[Das-Hacon 23, 24, 25]

[Das-Hacon-Poon 24]

[Das-Ou 24, 26]

[Das-Hacon-Yonez 23]

[BCHM10]

+ strongly $\mathbb{Q}$ -factorial

Thm (MMP)_n: $(X, B + \beta)$ cpt Kähler gbt pair

$\Rightarrow$ We can run $(K_X + B + \beta_X)$ -MMP with scaling of a Kähler form and it terminates with

(1) Mori fiber space, if $K_X + B + \beta_X$ is not pre-eff

(2) good minimal model, if $K_X + B + \beta_X$ is pre-eff
& $B + \beta_X$ is big

proof: Cone Thm [Hacon-Park 24, Hacon-Li-Xie 26]

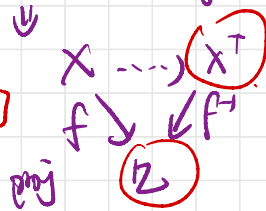
[Fuj22]

$\overline{NA}(X) =$



$\frac{K_X + B + \beta_X + W}{\Downarrow}$ nef

(Existence of flips [DPS-Hacon 23])



Kähler criterion [Hacon-Li-Xie 26]

Finiteness of minimal models.

Consider non-klt/lc g-pairs:

Thm (Contr)_n: $(X, B+\beta)$ opt Kähler g-pair, n -dim

• $B+\beta_X$ modified big

• $\alpha = K_X + B + \beta_X$ nef

• $\alpha|_W$ is "semi-ample", $W := \text{Nklt}(X, B+\beta)$

defined by $J(X, B+\beta)$
multiplier ideal

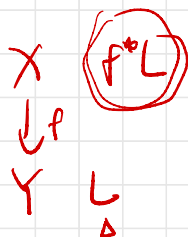
$\Rightarrow \alpha$ is "semi-ample"

analytic spaces

proper Mordell-Weil $f: X \rightarrow Y$

• $f(C) = \text{pt.}$ iff $\alpha \cdot C = 0$

• $f_* \mathcal{O}_X = \mathcal{O}_Y$

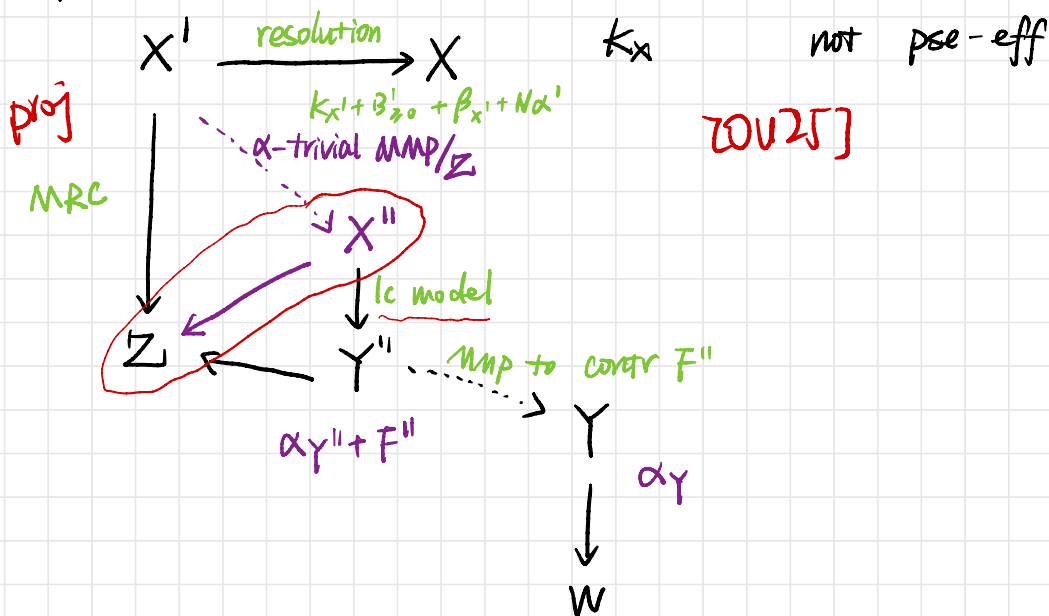


Idea of the proofs: α $\begin{cases} \text{big} \\ \text{not big} \end{cases}$

$$\text{Thm}(\text{MMP})_{n-1} + \text{Thm}(\text{BPF})_{n-1} \Rightarrow \text{Thm}(\text{BPF})_n, \text{ non-big}$$

$$\alpha = K_X + B + \beta_X \quad \text{not big}$$

$$\alpha' = K_{X'} + B' + \beta_{X'}$$



BPF

$$\text{Thm}(\text{Contr})_{n-1} \Rightarrow \text{Thm}(\text{Contr})_n, \text{ big}$$

$$X \xrightarrow{\log\text{-res}} X' \quad \alpha = K_X + B + \beta_X \quad \text{big}$$

$$\alpha' \equiv \overset{\text{SNC}}{\downarrow} E + y' \leftarrow \text{kähler}$$

$$\alpha' : X' \longrightarrow Z$$

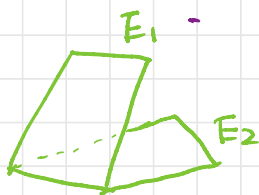
$$\alpha' \cdot C = 0$$

$$\Rightarrow C \subseteq E$$

inf (nE)

$$\xrightarrow{f_n} \text{Nkt}(X', B' + \beta + \lambda E + \lambda y') = LB' + \lambda E|_{\geq 0}$$

$\xrightarrow{f_n}$ Baby case : $\alpha = K_X + \omega \equiv E_1 + \frac{1}{2 \cdot 9} E_2 + y$



$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2 \cdot 9$$

$$\textcircled{1} 2\alpha|_{E_1} \equiv (K_X + \omega + E_1 + \frac{1}{2 \cdot 9} E_2 + y)|_{E_1}$$

$$f_1 : E_1 \rightarrow T_1 \quad \text{proj}$$

$$\textcircled{2} 3\alpha \equiv K_X + \omega + 2E_1 + \frac{2}{9} E_2 + 2y$$

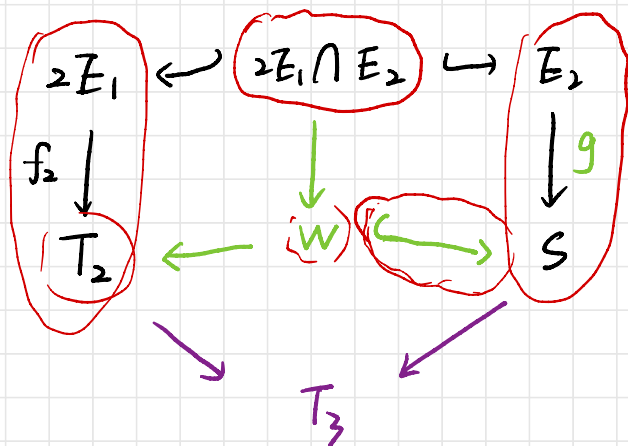
$$(2E_1) \rightarrow T_2$$

$$0 \rightarrow \bigoplus_{\mathbb{Z}E_1} (-E_1) \rightarrow \mathcal{O}_{\mathbb{Z}E_1} \rightarrow \mathcal{O}_{E_1} \rightarrow 0$$

$$\mathcal{O}_{E_1}(-E_1) \simeq$$

$$\boxed{f_{\infty} \mathcal{O}_{\mathbb{Z}E_1}} \rightarrow f_{\infty} \mathcal{O}_{E_1} \rightarrow 0$$

$$\textcircled{3} \quad 3.9 \alpha \mid_{E_2} \equiv \underline{Kx+W+2.9E_1+E_2+2.9y} \mid_{\hat{E}_2}$$



$$\begin{array}{ccc} A' & \xrightarrow{x \mapsto x^3} & A' \\ \downarrow x \mapsto (x+1)^2 & & \\ A' & & \end{array}$$

$$kZ(x+1)^2 \cap kZx^3 = k$$

$$\mathrm{Thm}(\mathrm{BPF})_n \Rightarrow \mathrm{Thm}(\mathrm{MMP})_n$$

$$\mathrm{Thm}(\mathrm{MMP})_n + \mathrm{Thm}(\mathrm{Contr})_{n-1} \Rightarrow \mathrm{Thm}(\mathrm{Contr})_n, \text{ not big}$$

$$\alpha' = K_{X'} + B' + \beta_{X'}$$

$$X' \xrightarrow{\text{alt modification}} (X, B + \beta) \leftarrow \text{not klt/lc}$$

$$\begin{aligned} & \text{--- } K_{X'} - \mathrm{MMP} \\ & \text{--- } K_{X'} + N\alpha' - \mathrm{MMP} \end{aligned}$$

$$\alpha = K_X + B + \beta_X \quad \text{not big}$$

$$(X'', B'' + \beta) \geq W$$

$$\downarrow \text{MFS}$$

$$Y$$

$$\rightarrow Z$$

if W dann Y

$$\downarrow \downarrow$$

$$Z$$